Input-Output Control of Overhead Cranes

Jia Chang Huang
April 26, 2012
Model

Source: Petit and Rouchon, 2002.
Problem Statement

Variables:
\[ X(x, t) = \text{horizontal displacement} \]
\[ \rho(x) = \text{linear density distribution of cable} \]
\[ \tau(x) = \text{tension of cable} \]

Governing Equation:
\[ \rho(x) \frac{\partial^2 X}{\partial t^2} - \frac{\partial}{\partial x} \left( \tau(x) \frac{\partial X}{\partial x} \right) = 0 \]

Boundary Conditions:
\[ X(L, t) = u(t) \quad \text{at} \quad x = L \]
\[ m \frac{\partial^2 X}{\partial t^2} - \tau(0) \frac{\partial X}{\partial x} = 0 \quad \text{at} \quad x = 0 \]
Simplified Model

Approximations: \[ \rho(x) = \rho \]
\[ \tau(x) = mg + x\rho g \]

Governing Equation:
\[ \frac{\partial^2 X}{\partial t^2} - \frac{\partial}{\partial x} \left( \left( \frac{mg}{\rho} + xg \right) \frac{\partial X}{\partial x} \right) = 0 \]

Boundary Conditions:
\[ X(L, t) = u(t) \quad \text{at} \quad x = L \]
\[ m\frac{\partial^2 X}{\partial t^2} - \tau(0) \frac{\partial X}{\partial x} = 0 \quad \text{at} \quad x = 0 \]
Derivation 1

Take Laplace transform in time:

\[ y \frac{\partial^2 \hat{X}}{\partial y^2}(y, s) + \frac{\partial \hat{X}}{\partial y}(y, s) - ys^2 \hat{X}(y, s) = 0 \]

Substitute in \( z = isy \):

\[ z^2 \frac{\partial^2 \hat{X}}{\partial z^2}(z, s) + z \frac{\partial \hat{X}}{\partial z}(z, s) + z^2 \hat{X}(z, s) = 0 \]

Solution is a Bessel function.

\[ \hat{X}(x, s) = AJ_0 \left( 2is \sqrt{\frac{x + \frac{m}{\rho}}{g}} \right) + BY_0 \left( 2is \sqrt{\frac{x + \frac{m}{\rho}}{g}} \right) \]
Derivation 2

After applying boundary conditions and some math ...

\[ \hat{X}(x, s) = \frac{2}{\pi^2} \sqrt{\frac{m}{\rho g}} \int_0^\pi \int_0^\pi G(x, \theta, \phi) \cos \theta e^{-\delta(x, \theta, \phi)s} s \hat{y}(s) \, d\theta d\phi \]
\[ + \frac{2}{\pi^2} \frac{m}{\rho g} \int_0^\pi \int_0^\pi G(x, \theta, \phi) e^{-\delta(x, \theta, \phi)s} s^2 \hat{y}(s) \, d\theta d\phi \]
\[ + \frac{1}{\pi^2} \int_0^\pi \int_0^\pi e^{-\delta(x, \theta, \phi)s} \hat{y}(s) \, d\theta d\phi \]

where \( \delta(x, \theta, \phi) = 2 \sqrt{\frac{m}{\rho g}} \cos \theta + 2 \sqrt{\frac{x + m}{g}} \cos \phi \) and \( G(x, \theta, \phi) = \ln \left( \sqrt{\frac{p}{m} x + 1} \frac{\sin^2 \phi}{\sin^2 \theta} \right) \).
Taking the inverse Laplace transform gives the final answer.

\[
X(x, t) = \frac{2}{\pi^2} \sqrt{\frac{m}{\rho g}} \int_0^{\pi} \int_0^{\pi} G(x, \theta, \phi) \cos \theta \, \dot{y}(t - \delta(x, \theta, \phi)) \, d\theta d\phi \\
+ \frac{2}{\pi^2} \frac{m}{\rho g} \int_0^{\pi} \int_0^{\pi} G(x, \theta, \phi) \ddot{y}(t - \delta(x, \theta, \phi)) \, d\theta d\phi \\
+ \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} y(t - \delta(x, \theta, \phi)) \, d\theta d\phi
\]

where \( \delta(x, \theta, \phi) = 2\sqrt{\frac{m}{\rho g}} \cos \theta + 2\sqrt{\frac{x + \frac{m}{g}}{g}} \cos \phi \) and \( G(x, \theta, \phi) = \ln \left( \sqrt{\frac{\rho}{m}} x + 1 \frac{\sin^2 \phi}{\sin^2 \theta} \right) \)
Simulation in Matlab
Simulation

Calculated input trajectories for different values of $\frac{m}{\rho L}$
Discrete Model

Discretize spatial variable using central difference scheme:

\[
\frac{\partial X}{\partial x} \bigg|_j \approx \frac{X_{j+1} - X_{j-1}}{2\Delta x} \\
\frac{\partial^2 X}{\partial x^2} \bigg|_j \approx \frac{X_{j+1} - 2X_j + X_{j-1}}{\Delta x^2}
\]

Approximate PDE as an N-state state-space equation.

\[
\frac{d}{dt} \begin{pmatrix} X \\ \dot{X} \end{pmatrix} = \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix} \begin{pmatrix} X \\ \dot{X} \end{pmatrix} + Bu
\]
Future Work

- Find a stable discrete model for PDE
- Implement closed-loop feedback control using linearized discrete model
References