Optimal Highway Traffic Control using a Velocity-Cell Transmission Model

Alessandro Castagnotto
Nicholas Wong
Outline

- Problem Description and Motivation
- Cell Transmission Model
- Godunov Flux function
- Cell Transmission Model for Velocity
- Optimal Control Problem
- Flux as a minimization problem
- Conclusions and Outlook
Problem Description and Motivation

- Optimal ramp metering for highways using a velocity based model
- Density data is relatively difficult to measure
- Proliferation of GPS and accelerometer equipped smart phones makes velocity data of vehicles on road available in abundance
Cell Transmission Model

- Discrete approximation to the LWR PDE

\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(\rho(x, t))}{\partial x} = 0
\]

- Highway is divided into discrete cells (indexed by \(i\))

- Density of vehicles in a cell, \(\rho\) changes in relation to the flows in and out, \(G\) the Godunov flux function

\[
\rho_i(k + 1) = \rho_i(k) + G_i(k) - G_{i+1}(k)
\]

Has been shown to be equivalent for piecewise affine flux
Godunov Flux Function

- Our flux function is the Greenshield’s Flux
- Previous work shows that a velocity model will not be equivalent unless the relationship between velocity and density is affine
- Using the relationship $Q = v \rho$ we obtain the following fundamental diagrams
Cell Transmission Model for Velocity

- Combining above equations give the following velocity based model

\[ v_{i}^{n+1} = V \left( V^{-1}(v_{i}^{n}) + \frac{\Delta T}{\Delta x} \left[ \tilde{G}(v_{i-1}^{n}, v_{i}^{n}) - \tilde{G}(v_{i}^{n}, v_{i+1}^{n}) \right] \right) \]
Godunov Flux

- The Godunov flux is given by:

\[ q_i^n = \tilde{G}(v_i^n, v_{i+1}^n) = \begin{cases} 
\tilde{q}(v_{i+1}), & \text{if } v_c \geq v_{i+1} \geq v_i \\
\tilde{q}(v_c), & \text{if } v_{i+1} \geq v_c \geq v_i \\
\tilde{q}(v_i), & \text{if } v_{i+1} \geq v_i \geq v_c \\
\min\{\tilde{q}(v_i), \tilde{q}(v_{i+1})\}, & \text{if } v_i \geq v_{i+1}
\end{cases} \]
Optimal Control Problem

Minimize $TTT$

s.t. $v_i^0, i = 1, ..., I$

$\tilde{G}(v_{i-1}, v_0) = \tilde{G}_0$

$\tilde{G}(v_i, v_{i+1}) = \tilde{G}_i$

$\tilde{G}(v_i, v_{i+1}) = \text{Godunov flux}$

- This leads to a problem as our flux is governed by a set of if conditions
Flux as a Minimisation Problem

An equivalent formulation for the flux can be found by:

\[ \tilde{G}(v_i, v_{i+1}) = \min \left\{ \begin{array}{c} sgn(v_i - v_{i+1})q(v_i) \\ sgn(v_i - v_{i+1})q(v_{i+1}) \\ \varepsilon q_{\text{max}} \end{array} \right\} \]

\[ \varepsilon = \begin{cases} -1, \text{when } v_i \leq v_c \leq v_{i+1} \\ \geq 1, \text{for all other cases} \end{cases} \]
Flux as a Minimisation Problem

- Mixed integer non-linear program

Minimize

\[ J_{\tilde{G}} \]

Subject to

\[ J_{\tilde{G}} = \alpha - 2g + \varepsilon_1 + \varepsilon_2 \]

\[ \alpha \geq g \]

\[ \alpha \geq -g \]

\[ g \leq sgn(v_1 - v_2)q(v_1) \]

\[ g \leq sgn(v_1 - v_2)q(v_2) \]

\[ g \leq [3 - \varepsilon_1 - \varepsilon_2]q_{\text{max}} \]

\[ \varepsilon_1 \geq 2sgn(v_2 - v_c) \]

\[ \varepsilon_1 \geq 0 \]

\[ \varepsilon_2 \geq 2sgn(v_2 - v_1) \]

\[ \varepsilon_2 \geq 0 \]
Flux as a Minimisation Problem

Mixed integer program with lower bounds on \( v_1 \) & upper bounds on \( v_2 \)
Conclusions and Outlook

- We are able to correctly determine flux despite non-linearities in the problem
- Slack variables affect our cost
- For control purposes the cost formulation needs to be tweaked or to reformulate the problem such that slack variables are not required (such as Big-M formulation)
- Infeasibilities with solvers when using system dynamics
Additional Material

Velocities obtained from the optimization program

![Graph showing velocities](image-url)
Additional Material

First 50 velocities obtained from simulation with additional upper bound on $v_1$ & lower bound on $v_2$
Mixed integer program with upper bounds on $v_1$ & lower bounds on $v_2$
Godunov Flux

\[ v_i \leq v_{i+1} \leq v_c \Rightarrow \tilde{G}_i = Q(v_2) \]

- Current and next cell are both congested
- Cell \( i+1 \) is moving faster
- The flux in cell \( i \) can be equal to that in cell \( i+1 \)
Godunov Flux

\[ v_i \leq v_c \leq v_{i+1} \implies \tilde{G}_i = Q(v_c) \]

- Cell \( i \) is congested
- Cell \( i+1 \) is not
- Cell \( i \) can discharge at maximum rate
Godunov Flux

\[ v_c \leq v_i \leq v_{i+1} \implies \tilde{G}_i = Q(v_1) \]

- Both cells are uncongested
- All vehicles can flow from \( i \) to \( i+1 \)
Godunov Flux

\[ v_{i+1} \leq v_i \implies \tilde{G}_i = \min\{Q(v_1), Q(v_2)\}\]

- Cell \( i+1 \) is moving slower than cell \( i \)
- The amount that can flow into cell \( i+1 \) is upper bounded by the flux of cars outside cell \( i+1 \)
Cell Transmission Model for Velocity

- Including ramps (control variable)

\[ v_{i}^{n+1} = V \left( V^{-1}(v_{i}^{n}) + \frac{\Delta T}{\Delta x} [\tilde{G}(v_{i-1}^{n}, v_{i}^{n}) - \tilde{G}(v_{i}^{n}, v_{i+1}^{n}) - R_{i}^{n} + S_{i}^{n}] \right) \]