Numerical Schemes from the Perspective of Consensus
Exploring Connections between Agreement Problems and PDEs

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Characterizing Numerical Schemes through Consensus

- Given a general PDE, how can we analyze it from the p.o.v. of Linear Systems Theory?
- Particularly interested in parabolic and elliptic PDEs that involve the Laplace operator
  - \( Au_{xx} + Bu_{xy} + Cu_{yy} + \ldots = 0 \), where \( B^2 - 4AC \leq 0 \)
  - For example, heat equation: \( u_t = au_{xx} \)
- Why the Laplace operator?
  - Rich theory of graphs utilizing discrete Laplace matrix
  - Concise framework through which to understand many problems in discrete distributed control
- **Question:** which classes of PDE control problems can be studied as consensus problems and how can this framework help us?
Outline of this talk

- General overview of the objectives of the project
- Mathematical Preliminaries
- Two Motivating Problems
  - Consensus on Networks
  - Control of Spatially Varying Interconnected Systems
- Illustrative example studying a PDE system as a consensus problem
- Broader discussion of the framework and potential future directions
Outline

1. Preliminaries
2. Two Motivating Problems
3. Illustrative Example: Heat Equation
4. Generalizations
5. Going Forward
Numerical Analysis well-understood, lends itself to solution of “real” ODE/PDE systems

Simplest example: First-order Euler explicit forward difference (from Taylor series)

\[ \dot{u} = f(t, u) \]
\[ u_{t+h} = u_t + h(t, u_t) \]
How Robust is the Scheme?

- Stability: Backward or implicit difference schemes stable for any $h$
  - Stiff Systems[1]
- Accuracy vs. efficiency: take into account more terms, tighter granularity in step size
- Choice may depend on operating constraints, e.g., online or offline
Numerical Schemes as Linear Systems

For systems, linear discretization results in an LTI system:

\[ \dot{u} = Su + Q, \quad u, Q \in \mathbb{R}^n, \quad S \in \mathbb{R}^{n \times n} \]

Standard linear systems tests apply for stability.

- See if $S$ is Hurwitz, etc.
Some Working Definitions for Graphs

Definition
A **vertex** $v$ is a point in $n$ dimensional space.

Definition
An **edge** is an ordered pair $(v_1, v_2)$ where $v_i$ are vertices connected by a segment.

Definition
A **graph** $G = G(V, E)$ is a collection of vertices $V$ connected by the set of edges $E$. 
Graph Models[3]

Cycle

Randomly Generated Graph
The Graph Laplacian

Definition

The **degree** of a vertex is the number of edges of which it is a member.

Definition

A graph **Laplacian** is a square matrix defined as follows[2]:

\[
[L]_{ij} = \begin{cases} 
  d_i & : i = j \\
  -1 & : \exists e_{ij} = (v_i, v_j) \in E \\
  0 & : \text{otherwise}
\end{cases}
\]  \hspace{1cm} (1)
Preliminaries
Two Motivating Problems
Illustrative Example: Heat Equation
Generalizations
Going Forward

Laplacian Spectral Analysis

- Laplacian is symmetric positive semidefinite
- At least one eigenvalue at zero, with corresponding eigenvector $[1 \ldots 1]^T$
- Spectrum of Laplacian describes network dynamics
  - Fiedler eigenvalue[1]: smallest positive eigenvalue a measure of connectedness
The Agreement Problem

- Given a collection of agents that can communicate
- Characterize conditions required for agreement on a parameter
- Describe convergence properties, e.g., asymptotic, exponential, ...
Consensus Problem on Networks

- Given a collection of nodes, model their interaction by

\[
\dot{x} = -Lx, \text{ continuous case}
\]
\[
x(k+1) = (I - \epsilon L)x(k), \text{ discrete case}
\]

**Theorem**

*For a continuous-time system, global exponential consensus is reached with speed at least as fast as \( \lambda_2(L) \). For a discrete-time system, global exponential consensus is reached with speed at least as fast as \( 1 - \epsilon \lambda_2(L) \), provided that \( \epsilon < \frac{1}{d_{\text{max}}} \), where \( d_{\text{max}} \) is the maximum degree of any vertex of \( L \).*
Fascinating emergent behavior: group of nodes behaves like a continuum reaching a steady-state value

Applications[3]
- Formation control
- Coupled oscillators
- Flocking
- Small world networks
- Distributed sensor fusion
Control of Spatially Varying Systems[3]

- Spatially varying system
- Defines necessary and sufficient LMI conditions for stabilization
- Example application: discretization of the heat equation in two dimensions
- Our goal: understand discretization from the perspective of consensus
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Control of Spatially Varying Systems[4], II

Finite linear connection

Finite planar connection\(^1\)

\(^{1}\text{D’Andrea, Langbort, and Chandra, 2003.}\)
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Heat Equation on the Unit Circle

- Heat equation on the circle [4] parametrized by \( x \in [0, 1) \) with \( \theta = 2\pi x \): \( u_t = au_{xx} \)
- Separation of variables readily yields:
  \[
  u(x, t) = (T_0 * H_t)(x) = \sum_{n=-\infty}^{\infty} a_n e^{-4\pi^2 n^2 t} e^{2\pi jnx}
  \]
- \( H_t(x) \) is the heat kernel for the unit circle and \( a_n \) are the Fourier coefficients of \( T_0 \)
- Eigenvalues given by \( 4\pi^2 n^2 \)
Spatial Representation[3]

Circle\(^2\) (1D Periodic BC)

\(^2\)D’Andrea and Dullerud, 2003.
Three point stencil

- 1D Laplacian: second-order centered difference scheme.

\[(u_t)_i = \frac{a}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1})\]

- Resulting space discretization: first order linear ODE
- Same dynamics as the consensus problem!
Laplacian Graph Structure and Interpretation

To impose periodic boundary conditions[2], note that \( u_0 = u_N \). The system representations are

\[
-L = \frac{a}{\Delta x^2}
\begin{pmatrix}
-2 & 1 & 0 & \ldots & 0 & 1 \\
1 & -2 & 1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \vdots & \ddots & 1 & -2 & 1 \\
1 & 0 & \ldots & 0 & 1 & -2
\end{pmatrix}
\]

\[
\dot{u} = -Lu \\
u(k+1) = (I - \epsilon L)u(k).
\]
Using discrete Fourier transform, with the eigenfunctions $f(j)(x) = e^{lk_j i \Delta x}$, we get

$$\lambda(\phi_m) = \frac{2a}{\Delta x^2} (\cos \phi_m - 1),$$

with $\phi_m = lk_m \Delta x = \frac{2m\pi}{N}$, $m = 0, \ldots N - 1$. So the eigenvalues range in value from $-\frac{4a}{\Delta x^2}$ to zero. Let $a = 1$ and $\Delta x = \frac{1}{N}$. Then:

$$\lambda(L) = \frac{2}{\Delta x^2} \left( 1 - \cos \frac{2m\pi}{N} \right) \approx 4m^2 \pi^2.$$
Spectral Analysis, II

- Eigenvalues of discretization converge to eigenvalues of continuous operator.
- $\lambda_2(L) \approx 4\pi^2$
- Space-discretized ODE will reach globally exponential consensus value with a speed at least as great as $4\pi^2$.
- Standard ODE time integration methods chosen based on location of eigenvalues
- Key idea: The original PDE system, discretized, is exactly the consensus problem for a cycle topology.
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Different Schemes

- Use different discretizations
- 5, 7, 9 point stencils
  - Algebraic connectivity improves
  - Computational cost increases
  - For a large grid, stencil size will be less important
- Modified 3 point stencil: look two steps behind and ahead
  - Doesn’t have much effect on connectivity
Extensions to higher dimensions

- Same idea applicable to higher spatial order Laplacian terms
  \[ u_t = a \Delta u \]
- Apply higher order spatial discretizations
- 2D case: typical choice 5 or 9 point scheme
  \[ (u_t)_{i,j} = \frac{a}{\Delta x^2} (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) \]
Visual Representation[3]

Torus$^3$ (2D Periodic BC)

Stencil Grids

One spatial dimension  Two spatial dimensions
Classes of Systems Amenable to Consensus Graph Analysis

- Periodic Boundary Conditions
- Neumann Boundary Conditions
  - Can control through the boundaries
- Control terms can also be applied at each actuator: turns out that if there are at least two independent inputs (outputs), LTI system is controllable (observable).
- General LTI form:
  \[
  \dot{u} = -Lu + Gr
  \]
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Horizons

- Other Boundary Conditions: cannot be directly represented as graph Laplacian
  - Potential future work in this direction: how can these systems be understood as consensus problems?
  - Can use linear state feedback to get to Laplacian form, but may not be realistic
- Designing better interconnection structures through graph theory
  - Boyd et al.: Convex optimization to design fastest mixing Markov chains, minimize resistance in a network, etc.
Summary

- Many second-order PDEs can be analyzed in a consensus framework
- Consensus a useful paradigm for distributed control
- Example: Heat Equation
- Laplacian graph theory and consensus:
  - Facilitate better numerical schemes
  - Aid in understanding complex continuum dynamics
Additional Reading

Additional Reading, II

Questions

Thank You

Any Questions?