Real–time Estimation of Flow States in Open Channels via Lagrangian Sensing

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Outline

- Problem statement
- Mathematical model
  - 1D Saint–Venant equations
  - Linearization & Discretization
  - State–space model
- Estimation set–up
- Implementation and Numerical results
- Future work
Real-time estimation of flow states (average velocity and average stage)
- Based on a 1D model
- Measurements obtained from Lagrangian sensors (drifters)
Governing equations

1D Saint–Venant equations:

\[ T \frac{\partial H}{\partial t} + \frac{\partial (THV)}{\partial x} = 0 \]
\[ \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial H}{\partial x} + g(S_f - S_b) = 0 \]

Friction slope:

\[ S_f = \frac{m^2V|V|}{(H)^{4/3}} \]
Linearization of the PDEs

\[ V(x, t) = \tilde{V}(x) + v(x, t) \]
\[ H(x, t) = \tilde{H}(x) + h(x, t) \]

Backwater curves (steady state):

\[ \frac{d\tilde{V}(x)}{dx} = -\frac{\tilde{V}(x)}{\tilde{H}(x)} \frac{d\tilde{H}(x)}{dx} - \frac{\tilde{V}(x)}{T(x)} \frac{dT(x)}{dx} \]
\[ \frac{d\tilde{H}(x)}{dx} = \frac{S_b - \tilde{S}_f}{1 - F(x)^2} \]

Carrying out the following approximation:

\[ f(V, H) \approx f(\tilde{V}, \tilde{H}) + (f_V)_{(\tilde{V}, \tilde{H})} v(x, t) + (f_H)_{(\tilde{V}, \tilde{H})} h(x, t) \]
Linearization of the PDEs

Linearized PDEs:

\[ h_t + \tilde{H}(x)v_x + \tilde{V}(x)h_x + \alpha(x)v + \beta(x)h = 0 \]
\[ v_t + \tilde{V}(x)v_x + gh_x + \gamma(x)v + \eta(x)h = 0 \]

where

\[ \alpha(x) = \frac{d\tilde{H}}{dx} + \frac{\tilde{H}}{T} \frac{dT}{dx} \]
\[ \beta(x) = -\frac{\tilde{V}}{\tilde{H}} \frac{d\tilde{H}}{dx} - \frac{\tilde{V}(x)}{T(x)} \frac{dT(x)}{dx} \]
\[ \gamma(x) = 2gm^2 \frac{\tilde{V}|\tilde{V}|}{\tilde{H}^{\frac{7}{3}}} - \frac{\tilde{V}}{\tilde{H}} \frac{d\tilde{H}}{dx} - \frac{\tilde{V}(x)}{T(x)} \frac{dT(x)}{dx} \]
\[ \eta(x) = -\frac{4}{3}gm^2 \frac{\tilde{V}|\tilde{V}|}{\tilde{H}^{\frac{7}{3}}} \]
Discretization

Lax Diffusive scheme:

\[
\frac{\partial f}{\partial t} = \frac{f_{i+1}^{k+1} - \frac{1}{2}(f_{i+1}^{k} + f_{i-1}^{k})}{\Delta t}
\]

\[
\frac{\partial f}{\partial x} = \frac{(f_{i+1}^{k} - f_{i-1}^{k})}{2\Delta x}
\]

\[
h_{i+1}^{k+1} = \frac{1}{2}(h_{i+1}^{k} + h_{i-1}^{k})
\]

\[
- \frac{\Delta t}{4\Delta x}(\bar{H}_{i+1} + \bar{H}_{i-1})(v_{i+1}^{k} - v_{i-1}^{k})
\]

\[
- \frac{\Delta t}{4\Delta x}(\bar{V}_{i+1} + \bar{V}_{i-1})(h_{i+1}^{k} - h_{i-1}^{k})
\]

\[
- \frac{\Delta t}{2}(\alpha_{i+1}v_{i+1}^{k} + \alpha_{i-1}v_{i-1}^{k})
\]

\[
- \frac{\Delta t}{2}(\beta_{i+1}h_{i+1}^{k} + \beta_{i-1}h_{i-1}^{k})
\]
Discretization

\[ v_{i+1}^{k+1} = \frac{1}{2}(v_{i+1}^k + v_{i-1}^k) \]

\[ - \frac{\Delta t}{4\Delta x}(\bar{V}_{i+1} + \bar{V}_{i-1})(v_{i+1}^k - v_{i-1}^k) \]

\[ - \frac{g\Delta t}{2\Delta x}(h_{i+1}^k - h_{i-1}^k) \]

\[ - \frac{\Delta t}{2}(\gamma_{i+1} v_{i+1}^k + \gamma_{i-1} v_{i-1}^k) \]

\[ - \frac{\Delta t}{2}(\eta_{i+1} h_{i+1}^k + \eta_{i-1} h_{i-1}^k) \]

CFL condition:

\[ \frac{\Delta t}{\Delta x} |V + C| \leq 1 \]
State-space model

The state-space model:

\[ z(k + 1) = Az(k) + Bu(k) \]

where

\[ z(k) = (v_2^k, \ldots, v_1^k, h_2^k, \ldots, h_1^k)^T \]
\[ u(k) = (v_1^k, h_1^k)^T \]
Esimation set-up

Stochastic model:

\[ z_{k+1} = Az_k + Bu_k + w_k \]
\[ y_k = C_z z_k + e_k \]

where the process and measurement noises are zero-mean Gaussian noise and,

\[ E[w_k w_k^T] = Q_k \delta_{kl} \]
\[ E[e_k e_k^T] = R_k \delta_{kl} \]
\[ z_0 = \mathcal{N}(\bar{z}_0, P_0) \]
Kalman Filter

Notation:

\[ \hat{z}_k = E[z_k|y_0, \cdots, y_k] \]
\[ \hat{z}_{k-1} = E[z_k|y_0, \cdots, y_{k-1}] \]
\[ P_k^- = \Sigma_k|k-1 \]
\[ P_k = \Sigma_k|k \]

Time update:

\[ \hat{z}_k^- = A\hat{z}_{k-1} + Bu_k \]
\[ P_k^- = AP_{k-1}A^T + Q \]

Measurement update:

\[ K_k = P_k^- C_k^T (C_k P_k^- C_k^T + R)^{-1} \]
\[ \hat{z}_k = \hat{z}_k^- + K_k(y_k - C_k \hat{z}_k^-) \]
\[ P_k = (I - K_k C_k) P_k^- \]
Implementation & Numerical Results

Experiment area: Sacramento River

(Courtesy of Qingfang Wu)
Implementation & Numerical Results
Implementation & Numerical Results

- Start at 3:40 PM March 16, 2007
- A single drifter deployed
- Number of cells: 30
- Spatial step size = 30 m
- Temporal step size = 3 sec
- Experiment duration: 38 min
Boundary conditions:

Average Velocity (m/s)  

Average Stage (m)
Implementation & Numerical Results

Drifter positions recorded every 3 seconds:
Implementation & Numerical Results

The estimated and the true average velocity at the 5\textsuperscript{th} and 12\textsuperscript{th} cells
The estimated and the true average velocity at the 19\textsuperscript{th} and 25\textsuperscript{th} cells
Relative error:

\[ \text{error}(k) = \sqrt{\frac{\sum_{i=1}^{N_{celt}} (u_i^k - \hat{u}_i^k)^2}{\sum_{i=1}^{N_{celt}} (u_i^k)^2}} \]

A single drifter

A static sensor at the 6\textsuperscript{th} cell
Future Works

- Implementing the method in a real experiment in real time
- Application to complex interconnected networks of channels
- Estimation of other quantities of interest (salinity, etc)
- Decentralized communication architecture
- Active drifters