Lecture 6: branch and bound

- Reminder: graphical construction of the feasible set
- An example of solution of an IP with branch and bound
- Branch and bound: summary of subproblems
- A generic branch and bound algorithm

[Bertsimas and Tsitsiklis, Introduction to Linear Optimization, Chap. 11, sec. 11.2, pp. 485-490]
Optimum of the linear program is not integer

\[ \text{min: } x_1 - 2x_2 \]
\[ \text{s.t. } -4x_1 + 6x_2 \leq 9 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1, x_2 \text{ integer} \]

Branch and bound algorithm

\[ \text{Solve LP } \rightarrow (1.5, 3.5), Z^* = -3.5 \]

\[ \text{min: } x_1 - 2x_2 \]
\[ \text{s.t. } -4x_1 + 6x_2 \leq 9 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1, x_2 \text{ integer} \]

Problem P1: add constraint \( x_2 \geq 3 \)

\[ \text{Solve P0 } \rightarrow (1.5, 3.5), Z^* = -3.5 \]
\[ \text{P1: Add constraint } x_2 \geq 3 \]
\[ \text{Problem P1 infeasible} \]

\[ \text{min: } x_1 - 2x_2 \]
\[ \text{s.t. } -4x_1 + 6x_2 \leq 9 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1, x_2 \text{ integer} \]

Problem P1: discard

\[ \text{Solve P0 } \rightarrow (1.5, 3.5), Z^* = -3.5 \]
\[ \text{P1: Add constraint } x_2 \geq 3 \]
\[ \text{Problem P1 infeasible} \]
\[ \text{Discard P1} \]

\[ \text{min: } x_1 - 2x_2 \]
\[ \text{s.t. } -4x_1 + 6x_2 \leq 9 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1, x_2 \text{ integer} \]

Problem P2: add constraint \( x_2 \leq 2 \)

\[ \text{Solve P0 } \rightarrow (1.5, 3.5), Z^* = -3.5 \]
\[ \text{P1: Add constraint } x_2 \leq 3 \]
\[ \text{Problem P1 infeasible} \]
\[ \text{Discard P1} \]
\[ \text{P2: P0 + constraint } x_2 \leq 2 \]

\[ \text{min: } x_1 - 2x_2 \]
\[ \text{s.t. } -4x_1 + 6x_2 \leq 9 \]
\[ x_1 + x_2 \leq 4 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1, x_2 \text{ integer} \]
Problem P2: solve for optimal

\[
\begin{align*}
\text{min:} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Solve P0 \to (1.5,3.5), Z^*=-3.5
P1: Add constraint x2 \geq 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2 \geq 2
Solve P2 \to (3/4,2), Z^*=-3.25

Problem P2: fractional optimum

\[
\begin{align*}
\text{min:} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Solve LP \to (1.5,3.5), Z^*=-3.5
P1: Add constraint x2 \geq 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2 \geq 2
Solve P2 \to (3/4,2), Z^*=-3.25

Problem P3: add constraint x1 \leq 0

\[
\begin{align*}
\text{min:} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Solve LP \to (1.5,3.5), Z^*=-3.5
P1: Add constraint x2 \geq 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2 \geq 2
Solve P2 \to (3/4,2), Z^*=-3.25
P3: P2 + constraint x1 \leq 0
Solve P3 \to (0,3/2), Z^*=-3

Problem P3: fractional solution

\[
\begin{align*}
\text{min:} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Solve LP \to (1.5,3.5), Z^*=-3.5
P1: Add constraint x2 \geq 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2 \geq 2
Solve P2 \to (3/4,2), Z^*=-3.25
P3: P2 + constraint x1 \leq 0
Solve P3 \to (0,3/2), Z^*=-3
P4: P2 + constraint x2 \geq 1
Solve P4 \to (1,2), Z^*=-3

Problem P4: add constraint x2 \geq 1

\[
\begin{align*}
\text{min:} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Solve LP \to (1.5,3.5), Z^*=-3.5
P1: Add constraint x2 \geq 3
Problem P1 infeasible
Discard P1
P2: P0 + constraint x2 \geq 2
Solve P2 \to (3/4,2), Z^*=-3.25
P3: P2 + constraint x1 \leq 0
Solve P3 \to (0,3/2), Z^*=-3
P4: P2 + constraint x2 \geq 1
Solve P4 \to (1,2), Z^*=-3
Problem P4: add constraint $x_2 \geq 1$

\[
\begin{align*}
\text{min: } & \quad x_1 - 2x_2 \\
\text{s.t.: } & \quad -4x_1 + 6x_2 \leq 9 \\
& \quad x_1 + x_2 \leq 4 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_1, x_2 \text{ integer}
\end{align*}
\]

Problem P1 infeasible
Discard P1

P2: Add constraint $x_2 \geq 2$
Solve P2 $\rightarrow (3/4,2), Z^*= -3.25$

P3: Add constraint $x_1 \geq 0$
Solve P3 $\rightarrow (0,3/2), Z^*= -3$

P4: Add constraint $x_2 \geq 1$
Solve P4 $\rightarrow (1,2), Z^*= -3$

**Optimum is** $(1,2)$

**Terminate**

Branch and bound: summary

**P1**: $\varnothing$
\[Z^* = \varnothing\]

**P2**: $(3/4,2)$
\[Z^* = -3.25\]

**P3**: $(0,3/2)$
\[Z^* = -3\]

**P4**: $(1,2)$
\[Z^* = -3\]

**Overall optimum** is $(1.5, 2.5)$, $Z^* = -3$. 

Solve LP $\rightarrow (1.5, 2.5), Z^* = -3.5$
A generic branch and bound algorithm (min. problem)

- Get upper bound U (solving relaxed LP)
- Select an active subproblem Pi
- If subproblem infeasible, delete it, otherwise, compute optimum for this subproblem (called U)
- If optimum greater than U, delete subproblem
- If optimum smaller than U, obtain optimal solution to the subproblem, or break corresponding subproblem into further subproblems which are added to the list of active subproblems
- Stop when list of active problems is empty