Lecture 3½ : gradient refresher

- Definition of the gradient in 2D
- Definition of the gradient in nD
- Graphical interpretation of the gradient
- Interpretation for linear programs
- Application for integer programming

Illustration of the gradient in 2D
Illustration of the gradient in 2D

\[ \frac{\partial f(x, y)}{\partial y} \]

Illustration of the gradient in 2D

\[
\begin{pmatrix}
\frac{\partial f(x, y)}{\partial x} \\
\frac{\partial f(x, y)}{\partial y}
\end{pmatrix}
\]
**Illustration of the gradient in 2D**

**Definition of the gradient in 2D**

\[ \nabla f(x, y) = \left( \begin{array}{c} \frac{\partial f(x, y)}{\partial x} \\ \frac{\partial f(x, y)}{\partial y} \end{array} \right) \]

This is just a generalization of the derivative in two dimensions. This can be generalized to any dimension.

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**Multiple dimensions**

Everything that you have seen with derivatives can be generalized with the gradient.

For the descent method, \( f'(x) \) can be replaced by

\[ \nabla f(x, y) = \left( \begin{array}{c} \frac{\partial f(x, y)}{\partial x} \\ \frac{\partial f(x, y)}{\partial y} \end{array} \right) \]

In two dimensions, and by

\[ \nabla f(x_1, x_2, \ldots, x_i, \ldots, x_N) = \left( \begin{array}{c} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_i} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{array} \right) \]

in \( N \) dimensions.
Example of 2D gradient: MATLAB demo

The cost to buy a portfolio is:

\[ f(x_1, x_2, \cdots, x_i, \cdots, x_N) = x_1^2 \cdot (x_2 - 4)^3 + \sum_{i=3}^{N} x_i^2 \]

If you want to minimize the price to buy your portfolio, you need to compute the gradient of its price:

\[ \nabla f(x_1, x_2, \cdots, x_i, \cdots, x_N) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_i}, \cdots, \frac{\partial f}{\partial x_N} \right) \]

Graphical interpretation of the gradient

The gradient of a scalar field (function of multiple variables) is perpendicular to the isolines of this function.
Interpretation: for linear programs

Take your favorite linear program:

\[ \begin{align*}
\text{min:} & \quad c_1 x_1 + c_2 x_2 + \cdots + c_{N_C} x_N \\
\text{s.t.:} & \quad a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,j} x_j + \cdots + a_{1,N_C} x_N \leq b_1 \\
& \quad a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,j} x_j + \cdots + a_{2,N_C} x_N \leq b_2 \\
& \quad \vdots \\
& \quad a_{M,1} x_1 + a_{M,2} x_2 + \cdots + a_{M,j} x_j + \cdots + a_{M,N_C} x_N \leq b_M
\end{align*} \]

Cost function of the linear program reads:

\[ J = c_1 x_1 + c_2 x_2 + \cdots + c_{N_C} x_N \]

Or in compact form:

\[ J = c \cdot x \]

Interpretation for linear programs
Isolines for the cost

\[ \nabla f(x_1, x_2) = c \]
Application for integer programming

\nabla f(x, y)

Application for integer programming

\nabla f(x, y)
Application for integer programming

\[ \nabla f(x, y) \]

Application for integer programming

\[ \nabla f(x, y) \]